(or)

$$I = \sin(a)$$

(Q.E.D)

Shivam Sharma

Third solution. Let I be the integral that we have to calculate. With the change of variable $t = -x$ we obtain:

$$I = \int_a^{-a} \frac{\cos(-x)}{1+\pi^{-\frac{1}{x}}}(-dx) = \int_{-a}^a \frac{\cos x}{1+\frac{1}{\pi^{\frac{1}{x}}}}dx = \int_{-a}^a \frac{\pi^{\frac{1}{x}}\cdot \cos x}{1+\pi^{\frac{1}{x}}}dx = \int_{-a}^a \frac{\pi^{\frac{1}{t}}\cdot t}{1+\pi^{\frac{1}{t}}}dt.$$

We obtain

$$I + I = \int_{-a}^a \frac{\cos t}{1+\pi^{\frac{1}{t}}}dx + \int_{-a}^a \frac{\pi^{\frac{1}{t}}\cdot \cos t}{1+\pi^{\frac{1}{t}}}dt = \int_{-a}^a \cos t dt = \sin t\Big|_{-a}^a =$$

$$= \sin a - \sin(-a) = 2\sin a.$$

From $2I = 2\sin A$ we deduce that $I = \sin a$.

Marin Chirciu

Fourth solution. Let $I := \int_{-a}^a \frac{\cos t}{\pi^{1/t}+1}dt$. Since

$$\int_{-a}^a \frac{\cos t}{\pi^{1/t}+1}dt = [x := -t; dx = -dt] = \int_a^{-a} \frac{\cos x}{\pi^{-1/x}+1}(-dx) =$$

$$\int_a^{-a} \frac{\cos x}{\pi^{-1/x}+1}(-dx) = \int_{-a}^a \frac{\cos x}{\pi^{-1/x}+1}dx = \int_{-a}^a \frac{\pi^{1/x}\cos x}{1+\pi^{1/x}}dx$$

then

$$2I = \int_{-a}^a \frac{\cos t}{\pi^{1/t}+1}dt + \int_{-a}^a \frac{\pi^{1/t}\cos t}{1+\pi^{1/t}}dt = \int_{-a}^a \frac{\cos t\left(1+\pi^{1/t}\right)}{\pi^{1/t}+1}dt = \int_{-a}^a \cos t dt =$$

$$= 2\int_0^a \cos t dt = 2\left(\sin t\right)_0^a = 2\sin a.$$

Hence $I = \sin a$.

Arkady Alt

Fifth solution.

$$\int_{-a}^{a} \frac{\cos t}{\pi^{1/t}+1} dt = \int_{-a}^{0} \frac{\cos t}{\pi^{1/t}+1} dt + \int_{0}^{a} \frac{\cos t}{\pi^{1/t}+1} dt$$

$$\int_{-a}^{0} \frac{\cos t}{\pi^{1/t}+1} dt \underbrace{=}_{t=-x} \int_{0}^{a} \frac{\cos x}{\pi^{-1/x}+1} dx$$

so we get

$$\int_{-a}^{a} \frac{\cos t}{\pi^{1/t}+1} dt = \int_{0}^{a} \frac{\cos x}{\pi^{-1/x}+1} dx + \int_{0}^{a} \frac{\cos t}{\pi^{1/t}+1} dt = \int_{0}^{a} \cos x dx = \sin a$$

Paolo Perfetti

W8. (Solution by the proposer.) First, we claim that all numbers of the form $a = 3^{-p}, p \geq 1$ and of the form $b = 7^{-q}, q \geq 1$ are accumulation points of A. Indeed, to show that each point of the form $a = 3^{-p}, p \geq 1$ is an accumulation point of A, we consider an arbitrary neighborhood $N(a)$ of A, say $N(a) = (a-h, a+h)$, with $h > 0$. Now, we choose an integer m so that $7^{-m} \leq h/2$. It always exists and it is given by $m = \lceil -\log(n/2)/\log 7 \rceil$.Then the point $y = 3^{-p} + 7^{-m}$ will be in $N(a) \cap A$. Therefore, it follows that all points of the form $a = 3^{-p}, p \geq 1$ are accumulation points of A. A similar argument can be used to show that all points of the form $b = 7^{-q}, q \geq 1$ are accumulation points of A.

Since every neighborhood of zero will contain a point from A if p and q are large enough, and because $0 \notin A$, then A is not a closed set.

Finally, remains to show that A is not an open set. To do it, it is sufficient to find a single point in A which does not have a neighborhood all of whose points belong to A. Taking the point in A obtained by setting $p = q = 1$, that is $10/21$, clearly we have that it is the largest element of the set A, and so every neighborhood of it of the form $(10/210 - h, 10/21 + h)$ for some $h > 0$ will contain elements which are larger than $10/21$ that cannot be expressed in the form $3^{-p} + 7^{-q}$. As a consequence, the point $10/21$ lies A, but it is not an interior point. So, A is not an open set.

José Luis Díaz-Barrero